First order transition amplitudes of the de Sitter QED and dynamical particle creation

Ion I. Cotăescu [*] and Cosmin Crucean [†]

West University of Timişoara,

V. Parvan Ave. 4 RO-300223 Timişoara, Romania

July 29, 2011

Abstract

It is shown that on the de Sitter spacetime global quantum modes can be prepared by a global apparatus which can measure QED transitions as in special relativity. The difference is that the gravity eliminates the mass-shell constraints giving rise to QED transitions with non-vanishing amplitudes even in the first order of perturbations. These amplitudes vanish in the flat limit where the mass-shell constraints are restored. Of a special interest could be the first order amplitudes of a new effect of dynamical particle creation due to the expansion of the de Sitter universe.

Pacs: 04.62.+v

Keywords: de Sitter QED; quantum fields; transition amplitudes; particle creation.

[*]E-mail: cota@physics.uvt.ro

[†]E-mail: crucean@physics.uvt.ro

1 Introduction

The theory of quantum fields on curved spacetimes is faced with the sensitive problem of separating the subspaces of the particle and antiparticle quantum modes. In special relativity the Poincaré isometries leave these subspaces invariant determining thus a stable vacuum state. In general relativity the curved spacetimes could have no isometries such that the criterion of separating the particle and antiparticle modes may be local. Then the vacuum becomes unstable in the sense that the quantum modes of two different frames may be related through Bogoliubov transformations which mix the particles and antiparticles modes among themselves [1]. This is the mechanism of cosmological particle creation giving rise to local effects in *accelerated* frames as, for example, the Unruh [2, 3] and Gibbons-Hawking [4] ones which may be observed using the *local* particle detectors proposed by Urunh [2] and DeWitt [5]. However, in the special case when the curved manifold has isometries able to take over the role of the Poincaré symmetry we can define a *global* apparatus which *prepares* and *measures* simultaneously the quantum states as in the flat case, without supplemental local detectors [6]. Then the vacuum state prepared by this apparatus is stable and the isometry generators become the principal operators (or observables) of the quantum theory.

The theory of the fields with spin on local-Minkowskian curved manifolds has a specific structure since the spin half can be correctly defined only in orthogonal local (non-holonomic) frames [7, 8]. Therefore, the Lagrangian theory of the matter fields must be written in local frames assuming that this is tetrad-gauge covariant. In this framework the gauge group (of the Minkowski metric) $L_+^\uparrow \subset SO(1,3)$ and its universal covering group, $SL(2,\mathbb{C})$, become crucial since their (non-unitary) finite-dimensional representations *induce* the *covariant* representations according to which the matter fields transform under isometries [9]. The generators of these representations are the differential operators produced by the Killing vectors associated to isometries according to the generalized Carter and McLenagan formula [10, 9, 11]. Thus in curved spacetimes with isometries there are operators globally defined on the entire manifold, independent on the natural or local frames one uses. Among these operators that commute with those of the field equations one can select the sets of commuting operators able to define the quantum modes as common systems of eigenfunctions. With this method the quantum modes are *globally* defined on the entire manifold and the vacuum is stable being invariant under isometries. Then the quantization can be done in a *canonical* manner

like in special relativity as we have shown in Refs. [12]-[15]. We stress that our conjecture is closer to the genuine principles of quantum mechanics but this does not exclude or contradict the cosmological particle creation which is observed in accelerated frames using local detectors [3, 16].

Our method is helpful on the de Sitter spacetime where all the free field equations can be analytically solved while the $SO(1,4)$ isometries offer us a large collection of operators commuting with the operators of these equations. A special problem of this geometry is that the only time-like Killing vector is not time-like everywhere. This generates some doubts concerning the possibility of defining correctly the energy operator, the *in* and *out* fields and the scattering operator, S [17]. However, we have shown that there are no real impediments since the time-like Killing vector keeps this property everywhere *inside* the light-cone where an observer can perform physical measurements [18]. We remind the reader that in this geometry the light-cone domain of any observer remains behind his horizon such that the measurements can be done as in the flat case without to be affected by the presence of the observer's horizon. Thus quantum transitions can be measured and their transition amplitudes can be calculated using perturbations in terms of free quantum modes. Since all the basis-generators of the $so(1,4)$ algebra have a well-defined physical meaning [18], we can select various sets of commuting operators determining different sets of free quantum modes. A particular feature of this algebra is that the energy and momentum operators do not commute to each other and, consequently, the dispersion relation is absent and there are no mass-shells [6, 18]. This affects the measurements of the energy and momentum which are diagonal in different bases, called here the momentum and respectively energy representations. Under such circumstances, we derived the principal quantum modes of the free Dirac [12, 13], Proca [14] and Maxwell [15] fields for which we used the same method of canonical quantization with a stable vacuum state of the Bounch-Davies type [19]. Moreover we obtained the expressions of the retarded, advanced and causal propagators of all these fields that enable us to define the *in* and *out* fields of the de Sitter QED [12, 15].

We must specify that our results based on canonical quantization are different from those derived by other authors which developed either a de Sitter quantum field theory [20] or a quantum mechanics [21] in axiomatic manners, trying to avoid the difficulties of the canonical theory by exploiting the high symmetry of the de Sitter manifold. Thus, in Refs [21] the tetrad-gauge covariance is neglected focusing only on the usual linear representations of

the $SO(1,4)$ group. However, these representations do not play any role in our canonical approach where the the matter fields transform according to covariant representations [9, 18]. In other respects, we observe that the vector propagators of the canonical theory [14] are different from the maximally symmetric $SO(1,4)$ two-point functions of Refs. [20] even though these satisfy the same equations. Moreover, the propagator of the Dirac field we derived as a modes sum [12] was put in a closed form [22] which is no longer maximally symmetric. The explanation is that in our approach the propagators of the fields with spin get some factors produced by the $SL(2,\mathbb{C})$ gauge symmetry which induces the covariant representations of the isometry group.

According to the above arguments we believe that our approach is appropriate for studying QED effects on the de Sitter manifold using perturbations as in special relativity [26, 28]. We assume that the QED transitions are measured by the *same* global apparatus which prepares the free quantum states. Then the *in* and *out* asymptotic fields are free fields minimally coupled to gravity but without electromagnetic interaction. Any transition amplitude between *in* and *out* states has to be calculated by using the scattering operator S expanded in terms of *in* fields as in the flat case. However, the principal difference is that the de Sitter geometry changes during the infinite period of the $in-out$ transitions when the space of this manifold is expanding from zero to infinity. This could lead to dramatic consequences including new quantum effects due to the gravity of the background.

In this paper we should like to study such simple effects focusing only on the non-vanishing $in-out$ amplitudes in the *first order* of perturbations as predicted by the QED on the de Sitter expanding universe. These amplitudes are very interesting since they correspond to transitions which in the flat limit are forbidden by the mass-shell constraints. On the other hand, we know that in QED the non-vanishing first-order contributions are dominant, determining the principal properties of these amplitudes. Our main objective here is to show that on the de Sitter spacetime there are $in-out$ transitions whose first-order amplitudes are non-vanishing and can be expressed in closed analytical forms. This enables us to calculate a new dynamical effect of particle creation showing that the classical gravitational field can give rise to quantum matter during the expansion of the de Sitter universe. Notice that the idea of the space expansion generating quantum matter is very old [23] but was less studied so far either in the WKB limit [24] or in the quantum theory but with a wrong geometric context [25]. For this reason we hope

that the new results we present here may be of interest.

We start in the second section presenting a natural version of QED written in moving charts with the conformal time and diagonal gauge. The S operator defined as in the flat case is expanded in series of perturbations in the momentum representation using the free Dirac and Maxwell fields. In the next section the first order $in - out$ transition amplitudes are calculated in terms of hypergeometric functions. The section 4 is devoted to the amplitude of the particle creation from the QED vacuum. Concluding remarks are presented in the last section.

2 The de Sitter QED

The tetrad-gauge covariant quantum field theory on a curved manifold (M, g) can be built in any chart $\{x^\mu\}$ $(\mu, \nu, ... = 0, 1, 2, 3)$ where one chooses the components of the tetrad fields $e_{\hat{\mu}}$ and $\hat{e}^{\hat{\mu}}$ defining the local orthogonal frames and coframes. These fields are labeled by local indices (with hat) which are raised or lowered by the Minkowski metric $\eta = \text{diag}(1, -1, -1, -1)$ while for the natural indices we have to use the metric tensor $g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}}\hat{e}^{\hat{\alpha}}_{\mu}\hat{e}^{\hat{\beta}}_{\nu}$. The commutation rules of the vector fields $\hat{\partial}_{\hat{\nu}} = e^{\mu}_{\hat{\nu}}\partial_\mu$ define the Cartan coefficients as $[\hat{\partial}_{\hat{\mu}}, \hat{\partial}_{\hat{\nu}}] = C_{\hat{\mu}\hat{\nu}}^{\cdot\cdot\hat{\sigma}}\hat{\partial}_{\hat{\sigma}}$. The connection components of any matter field of arbitrary spin can be written in local frames using the Cartan coefficients and the basis-generators of the finite-dimensional representations of the gauge group $SL(2, \mathbb{C})$ transforming the components of the matter fields [9].

Assuming that the matter fields of the tetrad-gauge covariant theory are minimally coupled to the gravity of M we consider the QED action

$$S = \int d^4x\sqrt{g}\ (\mathcal{L}_D + \mathcal{L}_{em} + \mathcal{L}_{int})\ , \quad g = |\det(g_{\mu\nu})|\,. \tag{1}$$

The Lagrangian density of the free Dirac field ψ of mass m has the form [12],

$$\mathcal{L}_D = \frac{i}{2} : [\overline{\psi}\gamma^{\hat{\alpha}}D_{\hat{\alpha}}\psi - (\overline{D_{\hat{\alpha}}\psi})\gamma^{\hat{\alpha}}\psi] - m\overline{\psi}\psi :, \quad \overline{\psi} = \psi^+\gamma^0\,, \tag{2}$$

where the point-independent Dirac matrices $\gamma^{\hat{\mu}}$ satisfy $\{\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}\} = 2\eta^{\hat{\alpha}\hat{\beta}}$ and define the basis-generators $S^{\hat{\alpha}\hat{\beta}} = i[\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}]/4$ of the spinor representation of the $SL(2, \mathbb{C})$ group. The spin connections $\hat{\Gamma}_{\hat{\mu}} = \frac{i}{4}(C_{\hat{\mu}\hat{\nu}\hat{\lambda}} - C_{\hat{\mu}\hat{\lambda}\hat{\nu}} - C_{\hat{\nu}\hat{\lambda}\hat{\mu}})S^{\hat{\nu}\hat{\lambda}}$ give the covariant derivatives of the Dirac field in local frames $D_{\hat{\alpha}} = e^{\mu}_{\hat{\alpha}}D_\mu =$

$\hat{\partial}_{\hat{\alpha}} + \hat{\Gamma}_{\hat{\alpha}}$. The term corresponding to the free electromagnetic field A reads [15],

$$\mathcal{L}_{em} = -\frac{1}{4} : F_{\mu\nu} F^{\mu\nu} : , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{3}$$

while the fields of the interaction term,

$$\mathcal{L}_{int} = -e_0 : \overline{\psi}(x) \gamma^{\hat{\mu}} e^{\nu}_{\hat{\mu}}(x) A_\nu(x) \psi(x) : , \tag{4}$$

are coupled by the elementary electric charge e_0 corresponding to the $U(1)_{em}$ gauge. The notation :: indicates the normal ordering of the products of the quantum fields that holds only when the vacuum state is well-defined and stable [1].

In what follows we consider (M, g) be the de Sitter spacetime where we choose the moving chart $\{t, \vec{x}\}$ of *conformal* time, $t \in (-\infty, 0)$, Cartesian coordinates and the line element

$$ds^2 = \frac{1}{(\omega t)^2} \left(dt^2 - d\vec{x} \cdot d\vec{x} \right) . \tag{5}$$

The tetrad fields we use have only diagonal components,

$$e^0_0 = -\omega t \,, \quad e^i_j = -\delta^i_j \, \omega t \,, \quad \hat{e}^0_0 = -\frac{1}{\omega t} \,, \quad \hat{e}^i_j = -\delta^i_j \, \frac{1}{\omega t} \,. \tag{6}$$

Notice that here the notation ω stands for the Hubble constant of the de Sitter spacetime.

In these frames we have solved the field equations of the free fields minimally coupled to the de Sitter gravity in momentum representation and we separated the positive and negative frequency modes in accordance to the choice of the *conformal* vacuum which is of the Bunch-Davies type [19] and stable on M [1]. Therefore, the canonical quantization can be performed as in special relativity from which we can take over the well-known methods of the QED in flat spacetimes. We note that the de Sitter QED vacuum does not represent the state of minimal energy of the whole system since the classical gravitational field is still present in this state. Moreover, the classical field can change energy with the quantum system without any constraint since there the dispersion relation does not work.

The free Dirac equation can be analytically solved in the gauge (6) obtaining the momentum and energy representations with correct orthonormalization and completeness properties [12, 13]. The mode expansion in the

momentum representation,

$$\psi(t,\vec{x})=\int d^3p\sum_{\sigma}\left[U_{\vec{p},\sigma}(x)a(\vec{p},\sigma)+V_{\vec{p},\sigma}(x)a^{c\dagger}(\vec{p},\sigma)\right]\,, \tag{7}$$

is written in terms of the field operators, a and a^c, and the particle and antiparticle spinors of this basis, $U_{\vec{p},\sigma}$ and respectively $V_{\vec{p},\sigma}$, which depend on the momentum $\vec{p}$ and the helicity $\sigma=\pm 1/2$. The canonical quantization requires the electron $(a,a^\dagger)$ and positron $(a^c,a^{c\dagger})$ field operators to satisfy the non-vanishing anti-commutators [12]

$$\{a(\vec{p},\sigma),a^\dagger(\vec{p}\,',\sigma')\}=\{a^c(\vec{p},\sigma),a^{c\dagger}(\vec{p}\,',\sigma')\}=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}\,')\,. \tag{8}$$

In the standard representation of the Dirac matrices (with diagonal γ^0) the particle and antiparticle spinors read [12],

$$\begin{aligned}
U_{\vec{p},\sigma}(t,\vec{x}\,) &= iN(\omega t)^2\left(\begin{array}{c}\frac{1}{2}\,e^{\pi\mu/2}H^{(1)}_{\nu_-}(-pt)\,\xi_\sigma(\vec{p})\\ \sigma\,e^{-\pi\mu/2}H^{(1)}_{\nu_+}(-pt)\,\xi_\sigma(\vec{p})\end{array}\right)e^{i\vec{p}\cdot\vec{x}} &(9)\\
V_{\vec{p},\sigma}(t,\vec{x}\,) &= iN(\omega t)^2\left(\begin{array}{c}-\sigma\,e^{-\pi\mu/2}H^{(2)}_{\nu_-}(-pt)\,\eta_\sigma(\vec{p})\\ \frac{1}{2}\,e^{\pi\mu/2}H^{(2)}_{\nu_+}(-pt)\,\eta_\sigma(\vec{p})\end{array}\right)e^{-i\vec{p}\cdot\vec{x}}\,, &(10)
\end{aligned}$$

where $p=|\vec{p}|$, $H^{(1,2)}_{\nu_\pm}$ are the Hankel functions of indices $\nu_\pm=\frac{1}{2}\pm i\mu$ with $\mu=\frac{m}{\omega}$, while

$$N=\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{\pi p}{\omega}}\,, \tag{11}$$

is the normalization constant. The Pauli spinors of the helicity basis, $\xi_\sigma(\vec{p})$ and $\eta_\sigma(\vec{p})=i\sigma_2[\xi_\sigma(\vec{p})]^*$, fulfill

$$\vec{\sigma}\cdot\vec{p}\;\xi_\sigma(\vec{p})=2p\,\sigma\,\xi_\sigma(\vec{p})\,,\quad \vec{\sigma}\cdot\vec{p}\;\eta_\sigma(\vec{p})=-2p\,\sigma\,\eta_\sigma(\vec{p})\,, \tag{12}$$

where σ_i are the Pauli matrices. The particle spinors have the form

$$\xi_{\frac{1}{2}}(\vec{p})=\sqrt{\frac{p_3+p}{2p}}\left(\begin{array}{c}1\\ \frac{p_1+ip_2}{p_3+p}\end{array}\right)\,,\quad \xi_{-\frac{1}{2}}(\vec{p})=\sqrt{\frac{p_3+p}{2p}}\left(\begin{array}{c}\frac{-p_1+ip_2}{p_3+p}\\ 1\end{array}\right)\,, \tag{13}$$

and satisfy the following properties

$$\sum_\sigma \xi_\sigma(\vec{p})\,\xi_\sigma(\vec{p})^+=1_{2\times 2}\,,\quad \sum_\sigma \sigma\,\xi_\sigma(\vec{p})\,\xi_\sigma(\vec{p})^+=\frac{\vec{\sigma}\cdot\vec{p}}{p} \tag{14}$$

Similar properties can be deduced for the anti-particle spinors η_σ associated to ξ_σ.

The free Maxwell field on the de Sitter spacetime can be canonically quantized in the Coulomb gauge (with $A_0 = 0$ and $\partial_i A_i = 0$) allowing the mode expansion [15]

$$\vec{A}(x) = \int d^3k \sum_\lambda \left[\vec{w}_{\vec{k},\lambda}(x)\alpha(\vec{k},\lambda) + \vec{w}_{\vec{k},\lambda}(x)^* \alpha^\dagger(\vec{k},\lambda) \right] , \tag{15}$$

in momentum representation where the modes functions $\vec{w}_{\vec{k},\lambda}$ depend on the momentum $\vec{k}$ and helicity $\lambda = \pm 1$. The photon field operators $(\alpha, \alpha^\dagger)$ satisfy the canonical commutation relations

$$[\alpha(\vec{k},\lambda), \alpha^\dagger(\vec{k}\,',\lambda')] = \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}\,') \,. \tag{16}$$

The solutions of the free Maxwell equation in momentum representation are [15]

$$\vec{w}_{\vec{k},\lambda}(t,\vec{x}\,) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2k}}\, e^{-ikt+i\vec{k}\cdot\vec{x}}\, \vec{\varepsilon}_\lambda(\vec{k}) \,, \tag{17}$$

where $k = |\vec{k}|$. The polarization vectors $\vec{\varepsilon}_\lambda(\vec{k})$ in the Coulomb gauge must be orthogonal to the momentum direction, $\vec{k}\cdot\vec{\varepsilon}_\lambda(\vec{k}) = 0$, for any polarization $\lambda = \pm 1$. In general, the polarization vectors have c-number components which must satisfy [28]

$$\begin{aligned} \vec{\varepsilon}_\lambda(\vec{k}) \cdot \vec{\varepsilon}_{\lambda'}(\vec{k})^* &= \delta_{\lambda\lambda'} \,, & (18)\\ \sum_\lambda \varepsilon_\lambda(\vec{k})_i\, \varepsilon_\lambda(\vec{k})_j^* &= \delta_{ij} - \frac{k^i k^j}{k^2} \,. & (19) \end{aligned}$$

Here we consider only the *circular* polarization and, therefore, we define the polarization vectors $\vec{\varepsilon}_{\pm 1}(\vec{k}) = \frac{1}{\sqrt{2}}(\mp\vec{e}_1 - i\vec{e}_2)$, in the three-dimensional orthogonal local frame $\{\vec{e}_i\}$ where $\vec{k} = k\vec{e}_3$.

The Fock space can be constructed now by using the above field operators and bearing in mind that the QED vacuum state $|0\rangle$ is a stable one obeying $a(\vec{p},\sigma)|0\rangle = a^c(\vec{p},\sigma)|0\rangle = \alpha(\vec{k},\lambda)|0\rangle = 0$. The creation operators give rise to the other states of the Fock space according to the standard procedure. For example, a state with n_1 electrons, n_2 positrons and n_3 photons is denoted by

$$|\, n_1(\vec{p},\sigma,-); n_2(\vec{p}\,',\sigma',+); n_3(\vec{k},\lambda)\rangle = a^\dagger(\vec{p},\sigma)^{n_1} a^{c\dagger}(\vec{p}\,',\sigma')^{n_2} \alpha^\dagger(\vec{k},\lambda)^{n_3}|0\rangle \,. \tag{20}$$

The vacuum and all the states with different number of particles constitute the momentum basis of the Fock space.

In the presence of the classical gravitational field the reduction formalism [26] works as in the flat case if the free fields are quantized canonically [27]. This enables us to study transition amplitudes,

$$\langle out;\beta....|in;\alpha...\rangle = \langle in;\beta....|S|in;\alpha...\rangle\,, \tag{21}$$

between an *in* state at $t = -\infty$ and an *out* state at $t = 0$. We note that in the moving chart with the proper time $\hat{t} = -\frac{1}{\omega}\ln(-\omega t)$ these states are defined as usual for $\hat{t} \to -\infty$ and $\hat{t} \to \infty$ respectively. These amplitudes have to be calculated using the perturbation method based on the expansion of the operator S in terms of *in* free fields. Recent results indicates that the definitions of the *in* and *out* fields as well as the mechanisms of the reduction formalism and perturbation procedure are similar to those of the flat case [27]. For this reason we can use the standard expansion

$$S = T\,\exp\left[-i\int d^4x\sqrt{g}\,\mathcal{L}_{int}(\psi_{in}, A_{in})\right] \tag{22}$$

where the operator products are put in the chronological order with respect to the conformal time t. The *in* fields are free fields which can be identified with those defined by Eqs. (7) and (15), $\psi_{in} = \psi$ and $A_{in} = A$. Replacing then in Eq. (4) the quantities $\sqrt{g} = (\omega t)^{-4}$ and $\gamma^{\hat{\mu}} e^{\nu}_{\hat{\mu}}(x) A_{\nu}(x) = -\omega t\,\vec{\gamma}\cdot\vec{A}(x)$, we arrive at the final formula

$$S = T\,\exp\left[-ie_0\int \frac{d^4x}{(\omega t)^3}\;:\overline{\psi}(x)\,\vec{\gamma}\cdot\vec{A}(x)\,\psi(x):\right] \tag{23}$$

which can be used in applications.

3 First order transition amplitudes

The QED transitions which have non-vanishing amplitudes in the first order of perturbations involve only three particles, a photon and two Dirac particles, which can appear in the *in* or *out* states. The allowed transitions are: one electron (e^-) or positron (e^+) emitting or absorbing one photon (γ), one photon pair creation or annihilation ($\gamma \to e^- + e^+$ and $e^- + e^+ \to \gamma$), the creation from the QED vacuum (*vac*) of the triplet $e^- + e^+ + \gamma$ and the annihilation of this triplet into the same vacuum. The transition $vac \to e^- + e^+ + \gamma$

represents a new *dynamical* effect of particle creation whose mechanism differs from that of the Unruh effect [2] which may be considered as being rather of a kinetic origin. The argument is that our effect is due to the QED interaction and disappears in the limit of the free fields ($e_0 \to 0$) where the Unruh effect is active.

The first order amplitudes have simple forms depending only on two spinors U and V and the photon mode function $\vec{w}$. The photon emission $e^- \to e^- + \gamma$ has the amplitude

$$\langle \vec{p}\,', \sigma', -; \vec{k}, \lambda |\, S_1 \,| \vec{p}, \sigma, - \rangle$$
$$= -ie_0 \int \frac{d^4x}{(\omega t)^3} \overline{U}_{\vec{p}\,',\sigma'}(x)\, \vec{\gamma} \cdot \vec{w}_{\vec{k},\lambda}(x)^* U_{\vec{p},\sigma}(x)\,. \quad (24)$$

When the photon is emitted by a positron we have to replace $U_{\vec{p}\,',\sigma'} \to V_{\vec{p},\sigma}$ and $U_{\vec{p},\sigma} \to V_{\vec{p}\,',\sigma'}$. Moreover, if we replace $\vec{w}^* \to \vec{w}$ we obtain the amplitudes of the transitions $e^- + \gamma \to e^-$ and $e^+ + \gamma \to e^+$ in which a photon is absorbed. In the cases of the pair creation, $\gamma \to e^- + e^+$, and annihilation, $e^- + e^+ \to \gamma$, we find the related amplitudes

$$\langle \vec{p}, \sigma, -; \vec{p}\,', \sigma', + |\, S_1 \,| \vec{k}, \lambda \rangle = -\langle \vec{k}, \lambda |\, S_1 \,| \vec{p}, \sigma, -; \vec{p}\,', \sigma', + \rangle^*$$
$$= -ie_0 \int \frac{d^4x}{(\omega t)^3} \overline{U}_{\vec{p},\sigma}(x)\, \vec{\gamma} \cdot \vec{w}_{\vec{k},\lambda}(x) V_{\vec{p}\,',\sigma'}(x)\,. \quad (25)$$

The dynamical particle creation was studied for the first time in [25] where the total amplitude was calculated between an *in* state at $t \to -\infty$ and the *out* state at $t \to \infty$. This means that the conformal time t covers the expansion period $(-\infty, 0]$ followed by a contraction, for $t \in [0, \infty)$. In this way the contraction cancels the effects due to the expansion, vanishing thus the total transition amplitude [25]. In our opinion, the contributions of the expansion and contraction periods must be treated separately considering the *out* state at $t = 0$ in the scenario of the expanding universe. Consequently, we find that the transitions $vac \to e^- + e^+ + \gamma$ and $e^- + e^+ + \gamma \to vac$ have non-vanishing amplitudes which can be derived by replacing $\vec{w} \to \vec{w}^*$ in Eq. (25).

All these amplitudes can be calculated according to Eqs. (9), (10) and (17). After a few manipulation we obtain

$$\langle \vec{p}\,', \sigma', -; \vec{k}, \lambda |\, S_1 \,| \vec{p}, \sigma, - \rangle$$

$$= -i\frac{e_0}{16\sqrt{\pi}}\sqrt{\frac{pp'}{k}}\,\delta^3(\vec{p}-\vec{p}\,'-\vec{k})\,\xi^+_{\sigma'}(\vec{p}\,')\,\vec{\sigma}\cdot\vec{\varepsilon}_\lambda(\vec{k})^*\xi_\sigma(\vec{p})$$
$$\times\left[\mathrm{sign}(\sigma)\,I_+^{(2,1)}(p,p',-k)+\mathrm{sign}(\sigma')\,I_-^{(2,1)}(p,p',-k)\right], \tag{26}$$
$$\langle\vec{p},\sigma,-;\vec{p}\,',\sigma',+|\,S_1|\,\vec{k},\lambda\rangle$$
$$= -i\frac{e_0}{16\sqrt{\pi}}\sqrt{\frac{pp'}{k}}\,\delta^3(\vec{p}+\vec{p}\,'-\vec{k})\,\xi^+_{\sigma}(\vec{p})\,\vec{\sigma}\cdot\vec{\varepsilon}_\lambda(\vec{k})\,\eta_{\sigma'}(\vec{p}\,')$$
$$\times\left[e^{\pi\mu}I_+^{(2,2)}(p,p',k)-\mathrm{sign}(\sigma\sigma')\,e^{-\pi\mu}I_-^{(2,2)}(p,p',k)\right], \tag{27}$$

where we denote

$$I_\pm^{(a,b)}(p,p',q)=\int_0^\infty ds\, s\, H^{(a)}_{\nu_\pm}(sp)H^{(b)}_{\nu_\pm}(sp')\,e^{iqs}\,,\quad a,b=1,2\,, \tag{28}$$

the time integrals of Hankel functions in the new variable $s=-t$. The obvious properties,

$$I_\pm^{(a,b)}(p,p',q) = I_\pm^{(b,a)}(p',p,q)\,,\quad a,b=1,2\,, \tag{29}$$
$${I_\pm^{(1,1)}(p,p',q)}^* = I_\mp^{(2,2)}(p,p',-q)\,, \tag{30}$$
$${I_\pm^{(1,2)}(p,p',q)}^* = I_\mp^{(2,1)}(p,p',-q)\,, \tag{31}$$

indicate that only two types of integrals are independent. Therefore we have nothing to lose if we restrict ourselves to study only the integrals $I_\pm^{(2,1)}$ and $I_\pm^{(2,2)}$ which are involved in the structure of our amplitudes.

The next step is to evaluate these integrals by expanding them in sums of integrals of J-functions as it results from Eqs. (51) and (52). Thus we obtain

$$I_\pm^{(2,1)}(p,p',q) = \frac{1}{\cosh^2\pi\mu}\left\{A_\pm(p,p',q)+C_\pm(p,p',q)\right.$$
$$\left.-ie^{\mp\pi\mu}B_\pm(p,p',q)+ie^{\pm\pi\mu}B_\pm(p',p,q)\right\}, \tag{32}$$
$$I_\pm^{(2,2)}(p,p',q) = \frac{1}{\cosh^2\pi\mu}\left\{e^{\mp2\pi\mu}A_\pm(p,p',q)-C_\pm(p,p',q)\right.$$
$$\left.+ie^{\mp\pi\mu}[B_\pm(p,p',q)+B_\pm(p',p,q)]\right\}, \tag{33}$$

where the new integrals

$$A_\pm(p,p',q) = \int_0^\infty ds\, s\, J_{\nu_\pm}(sp)J_{\nu_\pm}(sp')\,e^{iqs}$$

$$
\begin{aligned}
&= \frac{i}{\pi}\frac{q}{(pp')^{\frac{3}{2}}}\frac{d}{dz}\,Q_{\pm i\mu}[z-\mathrm{sign}(q)i0]\,, \qquad (34)\\
C_\pm(p,p',q) &= \int_0^\infty ds\, s\, J_{-\nu_\pm}(sp)J_{-\nu_\pm}(sp')\, e^{iqs}\\
&= \frac{i}{\pi}\frac{q}{(pp')^{\frac{3}{2}}}\frac{d}{dz}\,Q_{\mp i\mu-1}[z-\mathrm{sign}(q)i0]\,, \qquad (35)
\end{aligned}
$$

can be calculated straightforwardly using Eq.(53) while the integrals with indices of opposite signs,

$$
\begin{aligned}
B_\pm(p,p',q) &= \int_0^\infty ds\, s\, J_{\nu_\pm}(sp)J_{-\nu_\pm}(sp')\, e^{iqs} = -\frac{(1\mp 2i\mu)\cosh\pi\mu}{\pi k^2(1+4\mu^2)}\left(\frac{p}{p'}\right)^{\frac{1}{2}\pm i\mu}\\
&\times\ F_4\left(\frac{3}{2},1,\frac{3}{2}\pm i\mu,\frac{1}{2}\mp i\mu;\frac{p^2}{k^2+\mathrm{sign}(q)i0},\frac{{p'}^2}{k^2+\mathrm{sign}(q)i0}\right), \qquad (36)
\end{aligned}
$$

result from Eq. (54).

The functions $A_\pm$ and $C_\pm$ are expressed in terms of the Legendre functions of the second kind, $Q_\nu(z\pm i0)$, depending on the variable

$$
z=\frac{p^2+{p'}^2-k^2}{2pp'}\,, \qquad (37)
$$

which takes values in the domain $(-1,1)$ because of the momentum conservation in the amplitudes (26) and (38). Bearing in mind that the Legendre functions Q_ν have a branch cut in this domain we see that the small ϵ which assures the convergence of these integrals determines the analytic form the Legendre functions given in Appendix B. The functions $B_\pm$ have a more complicated structure depending on the Appell hypergeometric functions of double arguments F_4 [29]. Some technical difficulties could arise here because of these functions which are less studied so far. Nevertheless, we have all the ingredients we need for calculating the analytical expressions of these amplitudes.

4 The dynamical effect of particle creation

The examples we analyze now are the related amplitudes of the transitions $vac\to e^-+e^++\gamma$ and $e^-+e^++\gamma\to vac$ that read

$$
\langle\vec p,\sigma,-;\vec p\,',\sigma',+;\vec k,\lambda|\,S_1|\,0\rangle = -\langle 0|\,S_1|\,\vec p,\sigma,-;\vec p\,',\sigma',+;\vec k,\lambda\rangle^*
$$

$$= -i\frac{e_0}{16\sqrt{\pi}}\sqrt{\frac{pp'}{k}}\,\delta^3(\vec{p}+\vec{p}\,'+\vec{k})\,\xi^+_\sigma(\vec{p}\,)\,\vec{\sigma}\cdot\vec{\varepsilon}_\lambda(\vec{k})^*\,\eta_{\sigma'}(\vec{p}\,')$$
$$\times\left[e^{\pi\mu}I_+^{(2,2)}(p,p',-k)-\text{sign}(\sigma\sigma')\,e^{-\pi\mu}I_-^{(2,2)}(p,p',-k)\right] .\quad(38)$$

Simple kinetic parameters can be introduced in the orthogonal local frame $\{\vec{e}_i\}$ where $\vec{k}=-k\vec{e}_3$. In this frame we take the electron and positron momenta in the plane $(1,3)$ denoting their spherical coordinates as $\vec{p}=(p,\alpha,0)$ and $\vec{p}\,'=(p',\beta,\pi)$ where $\alpha,\,\beta\in(0,\pi)$. Then the momentum conservation gives the equations $k=p\cos\alpha+p'\cos\beta$ and $p\sin\alpha=p'\sin\beta$ from which we deduce

$$\frac{p}{k}=\frac{\sin\beta}{\sin(\alpha+\beta)}\,,\quad \frac{p'}{k}=\frac{\sin\alpha}{\sin(\alpha+\beta)}\,. \tag{39}$$

Moreover, from Eq. (37) we obtain $z=-\cos(\alpha+\beta)$ since the angle between $\vec{p}$ and $\vec{p}\,'$ is just $\alpha+\beta$.

The next step is to calculate the matrix elements

$$M_{\sigma,\sigma'}(\lambda)=\xi^+_\sigma(\vec{p}\,)\,\vec{\sigma}\cdot\vec{\varepsilon}_\lambda(\vec{k})^*\,\eta_{\sigma'}(\vec{p}\,') \tag{40}$$

corresponding to our geometry. We use Eqs. (13), the relation $\eta_\sigma=i\sigma_2\xi^*_\sigma$ and observe that in this case $\vec{\varepsilon}_{\pm1}(\vec{k})^*=\frac{1}{\sqrt{2}}(\pm\vec{e}_1+i\vec{e}_2)$ finding the following matrices

$$M(1) = \sqrt{2}\begin{pmatrix} -\cos\frac{\alpha}{2}\cos\frac{\beta}{2} & -\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\\ \sin\frac{\alpha}{2}\cos\frac{\beta}{2} & \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\end{pmatrix} \tag{41}$$
$$M(-1) = \sqrt{2}\begin{pmatrix} \sin\frac{\alpha}{2}\sin\frac{\beta}{2} & -\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\\ \cos\frac{\alpha}{2}\sin\frac{\beta}{2} & -\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\end{pmatrix} \tag{42}$$

for $\lambda=1$ and $\lambda=-1$ respectively.

Finally, we use Eqs. (34), (35) and (36) taking $q=-k$ for writing down the final expression

$$\langle\vec{p},\sigma,-;\vec{p}\,',\sigma',+;\vec{k},\lambda|\,S_1|\,0\rangle = -i\frac{e_0}{16\sqrt{\pi}}\,\frac{1}{k^{\frac{3}{2}}}\,\delta^3(\vec{p}+\vec{p}\,'+\vec{k})M_{\sigma,\sigma'}(\lambda)$$
$$\times\left\{\mathcal{F}_\mu(\alpha,\beta)+\mathcal{G}_\mu(\alpha,\beta)-\text{sign}(\sigma\sigma')\left[\mathcal{F}_{-\mu}(\alpha,\beta)+\mathcal{G}_{-\mu}(\alpha,\beta)\right]\right\}\,,\quad(43)$$

where the functions

$$\mathcal{F}_\mu(\alpha,\beta) = \frac{\mu(\mu-i)}{2\cosh^2\pi\mu\,\sinh\pi\mu}\,\frac{\sin^2(\alpha+\beta)}{\sin\alpha\sin\beta}\left[F\left(1-i\mu,2+i\mu;2;\cos^2\frac{\alpha+\beta}{2}\right)\right.$$

$$
-\sinh\pi\mu\, F\left(1-i\mu, 2+i\mu; 2; \sin^2\frac{\alpha+\beta}{2}\right)\Bigg] \tag{44}
$$

$$
\begin{aligned}
\mathcal{G}_\mu(\alpha,\beta) &= \frac{2\mu-i}{\pi(1+4\mu^2)}\,\frac{1}{\cosh\pi\mu}\,\frac{\sqrt{\sin\alpha\sin\beta}}{\sin(\alpha+\beta)}\left[\left(\frac{\sin\alpha}{\sin\beta}\right)^{\frac{1}{2}+i\mu}\right.\\
&\times F_4\left(\frac{3}{2},1,\frac{3}{2}+i\mu,\frac{1}{2}-i\mu;\frac{\sin^2\alpha}{\sin^2(\alpha+\beta)}+i0,\frac{\sin^2\beta}{\sin^2(\alpha+\beta)}+i0\right)\\
&\left.+(\alpha\longleftrightarrow\beta)\right]
\end{aligned} \tag{45}
$$

depend only on the angles α and β and the parameter μ.

It is remarkable that the above amplitudes depend on the fermion mass and the external gravity only on the parameter $\mu = \frac{m}{\omega}$. This parameter becomes very small under inflation when the Hubble constant ω is extremely large. This situation is well approximated by the limit of the amplitude (47) for $\mu\to 0$. Taking into account that in this limit (when $\nu_\pm \to \frac{1}{2}$) the Hankel functions are of the form (50) we can evaluate the integral

$$
\lim_{\mu\to 0} I_{\pm}^{(2,2)}(p,p',-k) = \frac{2i}{\pi}\frac{1}{\sqrt{pp'}}\frac{1}{k+p+p'-i0}\,, \tag{46}
$$

which leads to the amplitudes

$$
\begin{aligned}
&\lim_{\mu\to 0}\langle\vec{p},\sigma,-;\vec{p}\,',\sigma',+;\vec{k},\lambda|\,S_1|\,0\rangle\\
&\qquad = \frac{e_0}{8(\pi k)^{\frac{3}{2}}}\,\delta^3(\vec{p}+\vec{p}\,'+\vec{k})\delta_{\sigma,-\sigma'}M_{\sigma,-\sigma}(\lambda)\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}\,.
\end{aligned} \tag{47}
$$

These amplitudes are non-vanishing only if $\sigma' = -\sigma$. Thus, for $\lambda = 1$ we find two non-vanishing amplitudes proportional to

$$
\frac{e_0}{2(2\pi k)^{\frac{3}{2}}}\cos\frac{\alpha+\beta}{2}\times\left\{\begin{array}{ccc} \tan\frac{\alpha}{2} & \text{for} & \sigma=-\sigma'=-\frac{1}{2}\\ (-\tan\frac{\beta}{2}) & \text{for} & \sigma=-\sigma'=\frac{1}{2}\end{array}\right. \tag{48}
$$

Similar results written for $\lambda = -1$ show that all these amplitudes vanishes for $\alpha = \beta = 0$ when e^- and e^+ have parallel momenta in the same direction. However, whether e^- and e^+ have parallel momenta but in opposite directions, i. e. $(\alpha = \pi,\, \beta = 0)$ or $(\alpha = 0,\, \beta = \pi)$, we can not use the general formula (47) being forced to reconsider the momentum conservation. Let us take, for example, the *out* state with a photon having $\vec{k} = -k\,\vec{e}_3$ and

$\lambda = 1$, an electron of parameters $\vec{p} = p\,\vec{e}_3$ and $\sigma = \frac{1}{2}$ and a positron with $\vec{p}\,' = (k-p)\vec{e}_3$ provided $p > k$ and $\sigma' = -\frac{1}{2}$. Then the resulting amplitude for $\mu \to 0$ is proportional to

$$\frac{e_0}{2(2\pi)^{\frac{3}{2}}}\frac{1}{p\sqrt{k}}\,. \tag{49}$$

The conclusion is that under inflation the effect of particle creation is favored only when it produces pairs of fermions moving in opposite directions. Hereby we can ask if this phenomenon could be one of the mechanisms of separating the matter and antimatter between themselves.

5 Concluding remarks

We succeeded here to write down the transition amplitudes of the de Sitter QED which do not vanish in the first order of perturbations. It is remarkable that these transitions are due to the classical gravitational field which changes energy with the quantum matter eliminating thus the mass-shell constraints. All these transitions represent new quantum effects in external gravitational field which were less studied so far [25].

The above results were obtained using our gauge-covariant theory of quantum fields based on canonical quantization which exploits the isometries of the curved backgrounds. This allows us to correctly define the spin and to derive quantum modes as common systems of eigenfunctions of convenient sets of commuting operators. Moreover, in this approach the vacuum state is stable such that the canonical quantization, the development of the theory of interacting fields and the perturbation procedures can be done simply as in the flat case. For this reason we believe that our approach represents a good tool in calculating quantum transitions on the curved backgrounds which have isometries.

However, in the case of the de Sitter manifold, we did here only one step to a long way punctuated by many serious difficulties foreshadowed by the analytical forms of our amplitudes which are extremely complicated. We can imagine that the next orders of perturbations as well as the renormalization procedures will give rise to new technical difficulties in working with special functions and solving complicated integrals. The recent studies concerning the regularization of the photon [30] and electron-positron [22]

self-energy diagrams in the second order of perturbations confirm this perspective. Therefore, new mathematical methods are needed for solving these problems if we want to arrive to a strong theory of interacting fields on the de Sitter background, with complete Feynman rules and renormalization in any order.

Appendix A: Integrals of Bessel functions

The Bessel functions of index $\frac{1}{2}$ are elementary functions,

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}\,, \quad H^{(1)}_{\frac{1}{2}}(z) = -i\sqrt{\frac{2}{\pi z}}\, e^{iz}\,, \quad H^{(2)}_{\frac{1}{2}}(z) = i\sqrt{\frac{2}{\pi z}}\, e^{-iz}\,. \tag{50}$$

The first one helps us to evaluate the integrals (28) by replacing the exponential function in Eq. (28) and using the expansions of the Hankel functions in terms of Bessel functions J [29, 31],

$$H^{(1)}_{\nu_\pm}(z) = \frac{e^{\pm\pi\mu} J_{\nu_\pm}(z) - iJ_{-\nu_\pm}(z)}{\cosh(\pi\mu)}\,, \tag{51}$$

$$H^{(2)}_{\nu_\pm}(z) = \frac{e^{\mp\pi\mu} J_{\nu_\pm}(z) + iJ_{-\nu_\pm}(z)}{\cosh(\pi\mu)}\,. \tag{52}$$

We obtain thus two types of integrals which can be put in analytical forms.

The first integral [32],

$$\int_0^\infty dx\, x^{\frac{3}{2}}\, K_{\frac{1}{2}}(cx) J_\nu(ax) J_\nu(bx) = -\frac{1}{\sqrt{2\pi}} \frac{c^{\frac{1}{2}}}{(ab)^{\frac{3}{2}}} \frac{d}{du} Q_{\nu-\frac{1}{2}}(u)\,, \tag{53}$$

depends on the new variable u which obeys $2abu = a^2 + b^2 + c^2$. The second integral we consider [32],

$$\int_0^\infty dx\, x^{\frac{3}{2}}\, K_{\frac{1}{2}}(cx) J_\nu(ax) J_{-\nu}(bx) = \frac{\sin\pi\nu}{\sqrt{2\pi}\,\nu}\, c^{-\frac{5}{2}} \left(\frac{a}{b}\right)^\nu F_4\left(\frac{3}{2}, 1, 1+\nu, 1-\nu; -\frac{a^2}{c^2}, -\frac{b^2}{c^2}\right)\,, \tag{54}$$

is solved in terms of Appell hypergeometric functions F_4 depending on double arguments. Both these integrals are convergent for $\Re(c) > 0$. Therefore, for calculating Eqs. (34), (35) and (36) we are forced to replace $c \to \epsilon - iq$, introducing thus the usual $\epsilon > 0$ which finally tends to zero.

Appendix B: Legendre functions

The Legendre function of the second kind can be written as

$$Q_\nu(z \pm i0) = \frac{\pi}{2\sin\pi\nu}\left[e^{\mp i\pi\nu}P_\nu(z) - P_\nu(-z)\right] \tag{55}$$

for $-1 < z < 1$. The Legendre functions of the first kind,

$$P_\nu(z) = F\left(-\nu, 1+\nu; 1; \frac{1-z}{2}\right), \tag{56}$$

are analytic in this domain being represented by the usual Gauss hypergeometric function $F \equiv {}_2F_1$. Hereby we obtain the formula

$$\begin{aligned}\frac{d}{dz}Q_\nu(z \pm i0) &= \frac{\pi\nu(\nu+1)}{4\sin\pi\nu}\left[e^{\mp i\pi\nu}F\left(1-\nu, 2+\nu; 2; \frac{1-z}{2}\right)\right.\\ &\qquad \left. + F\left(1-\nu, 2+\nu; 2; \frac{1+z}{2}\right)\right]\end{aligned} \tag{57}$$

which helps us to calculate the functions $A_\pm$ (for $\nu = \pm i\mu$) and $C_\pm$ (taking $\nu = \mp i\mu - 1$).

References

[1] N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge 1982).

[2] W. G. Unruh, *Phys. Rev. D* **14** 879 (1976).

[3] L. C. B. Crispino, A. Haguchi and G. E. A. Matsas, *Rev. Mod. Phys* **80** (2008) 787, `arXiv:0710.5373`.

[4] G. W. Gibbons and S. W. Hawking, *Phys. Rev. D* **15** (1977) 2738 .

[5] B. S. DeWitt, *Quantum gravity: the new synthesis* in *General Relativity an Einstein centenary* survey ed. S. W. Hawking, W. Israel, (Cambridge University Press, Cambridge, 1979).

[6] I. I. Cotăescu, C. Crucean and A. Pop, *Int. J. Mod. Phys. A* **23** (2008) 2463.

[7] R. M. Wald, *General Relativity*, (Univ. of Chicago Press, Chicago and London, 1984)

[8] H. B. Lawson Jr. and M.-L. Michaelson, *Spin Geometry* (Princeton Univ. Press. Princeton, 1989).

[9] I. I. Cotăescu, *J. Phys. A: Math. Gen.* **33** (2000) 9177.

[10] B. Carter and R. G. McLenaghan, *Phys. Rev. D*, **19** (1979) 1093.

[11] I. I. Cotăescu, *Europhys. Lett.* **86** (2009) 20003 .

[12] I. I. Cotăescu, *Phys. Rev. D* **65** (2002) 084008.

[13] I. I. Cotăescu, *Phys. Rev. D* **60** (1999) 124006-010; I. I. Cotaescu, R. Racoceanu and C. Crucean, *Mod. Phys. Lett. A* **21** (2006) 1313; I. I. Cotaescu and C. Crucean, *Int. J. Mod. Phys. A* **23** (2008) 3707.

[14] I. I. Cotăescu, *GRG* **42** (2010) 861.

[15] I. I. Cotăescu and C. Crucean, *Progr. Teor. Phys.* **124** (2010) 1051, `arXiv:0806.2515 [gr-qc]`.

[16] H. Kodama, *Prog. Theor. Phys.* **63** (1980) 1217; R. Casadio, S. Ghiodini, A. Orlandi, C. Acquaviva and L Venzo, `arXiv:1011.3336`; P. Longhi and R. Soldati, `arXiv:1101.5976`.

[17] E. Witten, `hep-th/0106109`.

[18] I. I. Cotăescu, *GRG* **43**, 1639 (2011).

[19] T. S. Bunch and P. C. W. Davies, *Proc. R. Soc. Lond. A* **360** (1978) 117.

[20] B. Allen and T. Jacobson, *Commun. Math. Phys.* **103** (1986) 669; N. C. Tsamis and R. P. Woodard, *J. Math. Phys.* **48**, (2007) 052306, `gr-qc/0608069`.

[21] J-P. Gazeau and M.V. Takook, *J. Math. Phys.* **41** (2000) 5920; P. Bartesaghi, J. P. Gazeau, U. Moschella and M. V. Takook, *Class. Quantum. Grav.* **18** (2001) 4373; T. Garidi, J-P. Gazeau and M. Takook, *J.Math.Phys.* **44** (2003) 3838; J. P. Gazeau and M. Lachize-Rey, `arXiv:0802.3551`.

[22] J. F. Koksma and T. Protopec, *Class. Quantum Grav.* **26** (2009) 125003.

[23] E. Schrödinger, *Physica* **6** (1939) 899.

[24] L. Parker, *Phys, Rev. Lett.* **21** (1963) 562; L. Parker, *Phys. Rev.* **183** (1969) 1057; L. Parker, *Phys. Rev. D* **3** (1971) 346.

[25] L. I. Tsaregorodtsev, *Russian Phys. Journal* **41** (1989) 1028.

[26] S. Drell and J. D. Bjorken, Relativistic Quantum Fields (Me Graw-Hill Book Co., New York 1965)

[27] C. Crucean and R. Racoceanu, *Int. J. Mod. Phys. A* **23** (2008) 1.

[28] S. Weinberg, *The Quantum Theory of Fields* (Univ. Press, Cambridge, 1995).

[29] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, 1964)

[30] T. Prokopec, O. Törnkvist and R. P. Woodard, *Phys Rev. Lett.* **89** (2002) 101301; T. Prokopec, N. C. Tsamis and R. P. Woodard, *Phys. Rev. D* **79** (2008) 043523.

[31] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, 1922)

[32] I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press Inc., San Diego 1980).